

BOUNDS ON MULTIPLICITIES OF LAPLACE-BELTRAMI OPERATOR EIGENVALUES ON THE REAL PROJECTIVE PLANE

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ABSTRACT. The known upper bounds for the multiplicities of the Laplace-Beltrami operator eigenvalues on the real projective plane are improved for the eigenvalues with even indexes. Upper bounds for Dirichlet, Neumann and Steklov eigenvalues on the real projective plane with holes are also provided.

INTRODUCTION

Consider a closed surface Σ with a Riemannian metric g . Let

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

be Laplace-Beltrami operator eigenvalues (counting with multiplicities). Let $m(\Sigma, g, \lambda_i)$ denote the multiplicity of eigenvalue λ_i , i.e. the dimension of the eigenspace corresponding to λ_i .

Finding upper bounds for $m(\Sigma, g, \lambda_i)$ is one of classical problems of Spectral Geometry. These bounds are interesting from several points of view. For example, they play an important role in the study of metrics extremal for the Laplace-Beltrami eigenvalues, see e.g. [16].

In the present paper we consider the case of Σ being the real projective plane \mathbb{RP}^2 . The best known bound for the real projective plane

$$(1) \quad m(\mathbb{RP}^2, g, \lambda_i) \leq 2i + 3$$

was proven in the paper [14], see Theorem 5 below. Recently this bound was improved in the paper [15] for $i = 2$,

$$m(\mathbb{RP}^2, g, \lambda_2) \leq 6,$$

see Theorem 7 below.

The first goal of the present paper is to improve the bounds (1) for all even i and prove the following theorem.

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Theorem 1. *Let l be a positive integer, then*

$$m(\mathbb{R}P^2, g, \lambda_{2l}) \leq 4l + 1.$$

We also consider the case of a surface Σ with a boundary $\partial\Sigma$. In this case, let

$$0 \leq \lambda_0 \leq \lambda_1 \leq \dots$$

denotes the spectrum of the Laplace-Beltrami operator, given either Dirichlet or Neumann boundary condition on each connected component of the boundary $\partial\Sigma$.

We also consider Steklov eigenvalue problem on $(\Sigma, \partial\Sigma)$. Let ρ be a bounded non-negative function on $\partial\Sigma$ and σ a real number. Then a function u on Σ is called a Steklov eigenfunction with eigenvalue σ if

$$\begin{cases} \Delta u = 0 \text{ in } \Sigma, \\ \frac{\partial u}{\partial n} = \sigma \rho u \text{ on } \partial\Sigma, \end{cases}$$

where n denotes the unit outer normal on $\partial\Sigma$.

Let us consider the case where ρ is the density of an absolutely continuous Radon measure s on $\partial\Sigma$. Then the spectrum of Steklov problem is non-negative and discrete [1], so we denote it by

$$0 = \sigma_0 \leq \sigma_1 \leq \dots$$

Let us denote the multiplicity corresponding to the Steklov eigenvalue σ_i by $m(\Sigma, g, s, \sigma_i)$.

Let now $\mathbb{R}P_h^2$ denote $\mathbb{R}P^2 \setminus (\cup D_i^2)$, i.e. the real projective plane with a positive number of holes. Our second result then is the following theorem.

Theorem 2. *Let l be a positive integer and s be an absolutely continuous Radon measure on $\partial\mathbb{R}P_h^2$ with bounded density, then*

$$(2) \quad m(\mathbb{R}P_h^2, g, \lambda_{2l}) \leq 4l + 2,$$

$$(3) \quad m(\mathbb{R}P_h^2, g, s, \sigma_{2l}) \leq 4l + 2.$$

The proofs follow the technique from the paper [14] but uses a more refined topological argument at the last step.

Let us recall some already known results about upper bounds on multiplicities of eigenvalues.

Theorem 3 (Cheng [7]). *Let Σ be an oriented surface of genus γ . Then for any metric g one has*

$$m(\Sigma, g, \lambda_i) \leq \frac{(2\gamma + i + 1)(2\gamma + i + 2)}{2}.$$

Theorem 4 (Besson [4]). *Let Σ be an oriented surface of genus γ . Then for any metric g one has*

$$m(\Sigma, g, \lambda_i) \leq 4\gamma + 2i + 1.$$

Let Σ be a non-orientable surface of Euler characteristic $\chi(\Sigma)$. Then for any metric g one has

$$m(\Sigma, g, \lambda_i) \leq 4(1 - \chi(\Sigma)) + 4i + 2.$$

Theorem 5 (Nadirashvili [14]). *For any metric g on the sphere \mathbb{S}^2 , the real projective plane \mathbb{RP}^2 , the torus \mathbb{T}^2 , or the Klein bottle \mathbb{KL} the following inequalities hold,*

$$m(\mathbb{S}^2, g, \lambda_i) \leq 2i + 1,$$

$$m(\mathbb{RP}^2, g, \lambda_i) \leq 2i + 3,$$

$$m(\mathbb{T}^2, g, \lambda_i) \leq 2i + 4,$$

$$m(\mathbb{KL}, g, \lambda_i) \leq 2i + 3.$$

For any other surface Σ , i.e. for a surface Σ with $\chi(\Sigma) < 0$, with any metric g the following inequality holds,

$$m(\Sigma, g, \lambda_i) \leq 2i - 2\chi(\Sigma) + 3.$$

Theorem 6 (M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and Nadirashvili [9]). *Let Σ be a closed surface of genus 0. Then for any metric g the inequality*

$$m(\Sigma, g, \lambda_i) \leq 2i - 3$$

holds for $i \geq 3$.

Theorem 7 (Nadirashvili and Penskoi [15]). *The following upper bound for the multiplicity of the second eigenvalue of the Laplace-Beltrami operator on the projective plane holds for any metric g ,*

$$m(\mathbb{RP}^2, g, \lambda_2) \leq 6.$$

For a surface Σ with boundary $\partial\Sigma$ let us denote by $\bar{\Sigma}$ the closed (topological) manifold, obtained by collapsing each connected component of $\partial\Sigma$ into a point.

Theorem 8 (Karpukhin, Kokarev, and Polterovich [12]). *Let (M, g) be a compact Riemannian surface with a non-empty boundary, and μ be an absolutely continuous Radon measure on ∂M whose density is bounded. Then the multiplicity $m(M, g, \mu, \sigma_k)$ of a Steklov eigenvalue $\sigma_k(g, \mu)$ satisfies the inequalities*

$$(4) \quad m(M, g, \mu, \sigma_k) \leq 2(2 - \chi(\bar{M})) + 2k + 1,$$

$$(5) \quad m(M, g, \mu, \sigma_k) \leq 2(2 - \chi(M)) + k,$$

for all $k = 1, 2, \dots$

Besides, the latter inequality is strict for an even k .

Theorem 9 (Jammes [11]). *Let Σ be a compact surface with boundary, then for $k \geq 1$ the following inequality holds,*

$$(6) \quad m(\Sigma, g, \mu, \sigma_k) \leq k - 2\chi(\Sigma) + 3.$$

Theorem 10 (T. Hoffmann-Ostenhof, Michor, and Nadirashvili [10]). *Let $k \geq 3$. Then the multiplicity of the k -th eigenvalue λ_k for the Dirichlet problem on a planar simply-connected domain D satisfies the inequality*

$$m(D, \lambda_k) \leq 2k - 1.$$

Theorem 11 (Berdnikov [2]). *Let M be the surface with $\chi(\overline{M}) < 0$. Then*

$$\begin{aligned} m(M, g, \lambda_k) &\leq 2k - 2\chi(\overline{M}) + 3, \\ m(M, g, \text{vol}_{\partial M}, \sigma_k) &\leq 2k - 2\chi(\overline{M}) + 3. \end{aligned}$$

Let us also recall results concerning the relation between bounds on multiplicities of eigenvalues and the chromatic number.

Definition 1. *A chromatic number $\text{chr}(\Sigma)$ of a surface Σ is the maximal n such that one can embed in Σ the complete graph K_n on n vertices.*

Let us consider a Schrödinger operator $H = \Delta + V$. Let $\bar{m}_1(\Sigma)$ denote the supremum over all H of the multiplicity of the eigenvalue λ_1 on Σ .

Theorem 12 (Colin de Verdière [5]). *For any surface Σ , one has $\bar{m}_1(\Sigma) \geq \text{chr}(\Sigma) - 1$.*

For the four surfaces Σ with $\chi(S) \geq 0$, namely \mathbb{S}^2 , \mathbb{RP}^2 , \mathbb{T}^2 , \mathbb{KL} , one has $\bar{m}_1(\Sigma) = \text{chr}(\Sigma) - 1$.

This theorem leads to a natural conjecture that for any surface Σ one has $\bar{m}_1(\Sigma) = \text{chr}(\Sigma) - 1$. This conjecture was proved by Sévenec for some surfaces Σ using the following upper bound for $\bar{m}_1(\Sigma)$.

Theorem 13 (Sévenec [17]). *If $\chi(\Sigma) < 0$ then $\bar{m}_1(\Sigma) \leq 5 - \chi(\Sigma)$.*

It follows that the above conjecture holds for all surfaces Σ with $\chi(\Sigma) \geq -3$, the four new cases being $\mathbb{T}^2 \# \mathbb{T}^2$ and $\#n\mathbb{RP}^2$, $n = 3, 4, 5$.

Since $m(\Sigma, g, \lambda_1) \leq \bar{m}_1(\Sigma)$, these results provide interesting upper bounds on $m(\Sigma, g, \lambda_1)$.

Let us now compare Theorem 2 with the above mentioned results. The bound given by formula (2) seems to be a first bound of this kind for the projective plane with holes.

The comparison of the bound given in formula (3) with bounds (4), (5) and (6) gives a complicated answer. Bounds (4), (5) are linear in k , bound (4) has a better constant, bound (5) has a better asymptotic. Bound (3) improves bound (4) for \mathbb{RP}^2 by 1. Bound (6) improves bound (5) by 1.

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1. PRELIMINARIES

The standard relation between the original problem and the topology of the surface is well-known. Recall that for an eigenfunction f of the Laplacian, the connected components of $f^{-1}(\mathbb{R} \setminus \{0\})$ are called *nodal domains* of f and the set $f^{-1}(0)$ is called a *nodal graph* $\mathcal{N}(f)$ of f . We will denote the number of nodal domains of a (nodal) graph \mathcal{N} by $\mu(\mathcal{N})$. We will also use notation $\overline{\mathcal{N}}(f)$ for $\overline{\mathcal{N}(f) \cap \text{Int}(\Sigma)} \subset \overline{\Sigma}$. The term “graph” is justified by the following theorem.

Theorem 14 (Bers theorem, [3]). *For a Laplacian eigenfunction f and $x_0 \in M$ there exists an integer $n \geq 0$ and polar coordinates (r, θ) centered at x_0 such that the following formula holds*

$$f(x) = r^n(\sin(n\theta)) + O(r^{n+1}).$$

The Bers theorem implies that the nodal graph is, in fact, an embedded graph. It also implies that each eigenfunction has a well-defined order $\text{ord}(f, x) := n$ of vanishing at any point $x_0 \in M$. Note that the quotient of eigenfunctions f such that $\text{ord}(f, x) = n$ by those g with $\text{ord}(g, x) = n+1$ is contained in the span of $r^n(\sin(n\theta))$ and $r^n(\cos(n\theta))$ and hence at most 2-dimensional (if $n = 0$ then $\sin(n\theta) = \cos(n\theta)$ and the quotient is at most 1-dimensional). That implies that high-dimensional eigenspaces contain functions with high vanishing order at a given point.

Proposition 1. *Let U be an eigenspace of Laplacian with $\dim(U) \geq n$ and x be a point in M . Then the dimension of the linear subspace of functions $f \in U$ such that $\text{ord}(f, x) \geq k > 0$ (or, equivalently, x is a vertex of $\mathcal{N}(f)$ of $\deg \geq 2k$) is at least $1 + n - 2k$.*

The following theorem is crucial for the considered approach, it contains an upper bound for the number of nodal domains.

Theorem 15 (Courant nodal domain theorem, [8], [6], [13]). *Let Σ be a smooth manifold with smooth boundary $\partial\Sigma$ (possibly empty) and s be an absolutely continuous Radon measure on ∂M with bounded density. Consider either the Laplacian eigenvalue problem on M with the Dirichlet or Neumann boundary condition on each connected component of $\partial\Sigma$, or, if $\partial\Sigma \neq \emptyset$, the Steklov eigenvalue problem for the measure s . Then for each non-zero function f in the i -th eigenspace U_i of the considered spectral problem the number $\mu(\mathcal{N}(f))$ of its nodal domains is not greater than $i + 1$.*

The Theorems 1 and 2 follow immediately from the Courant nodal domain theorem and the following theorem, which we prove later in this paper.

Theorem 16. *Let λ be a real number and U be a linear space of functions on the closed surface $\Sigma = \mathbb{R}P^2$ or a surface with the boundary $\Sigma = \mathbb{R}P_h^2$. Suppose that every $f \in U$ is a Laplacian eigenfunction $\Delta f = \lambda f$. Suppose $\sup_{f \in U} (\mu(\mathcal{N}(f)))$ is not greater than an odd number $n = 2l + 1$. Let $i = n - 1$. Then $\dim(U) \leq 2n = 2i + 2$.*

Moreover, if $\Sigma = \mathbb{R}P^2$, i. e. $\partial\Sigma = \emptyset$, then this inequality is strict, i.e. $\dim(U) < 2n = 2i + 2$.

We will use a notion of a star fibration, which we explain now. Bers theorem tells us that the nodal graph of the eigenfunction f is diffeomorphic near x_0 to $2n$ rays in \mathbb{R}^2 emitting from 0 at equal angles between the adjacent lines. That makes it natural to consider a star fibration $E_M(2n)$ over $\text{Int}(M)$ introduced in paper [2]. It can be defined as follows. Recall that the spherisation $S(E)$ of a vector bundle E is the fiber bundle $(E \setminus \underline{0})/\mathbb{R}_{>0}$. In the case when E carries a positive-definite metric (i.e. tangent bundle of a Riemannian manifold), the spherisation $S(E)$ is isomorphic to the fiber bundle of the unit spheres in E . Now, consider the subset $E'_M(2n) \subset \prod^{2n} S(TM)$ such that the point $((x_1, v_1), \dots, (x_{2n}, v_{2n}))$ belongs to $E'_M(2n)$ iff all x_i are equal and v_i are representing equidistant rays in $T_x M$. Define $E_M(2n)$ as a quotient of $E'_M(2n)$ by the natural action of the permutation group S_n . The fiber $F_x(2n)$ at a point x of the fibration $E_M(2n)$ consists then of all $2n$ -stars in the tangent space $T_x M$, i.e. configurations of $2n$ rays (or equally n lines) in $T_x M$ with equal angles between adjacent lines.

In these terms the Bers theorem states that if $\text{ord}(f, x) = n$ then the nodal graph $\mathcal{N}(f)$ defines an element of $F_x(2n)$ which we denote by $s(\mathcal{N}(f), x)$.

2. FIRST BOUND

We start by proving Lemma 1 from paper [2] for our particular case $\Sigma = \mathbb{RP}^2$ or $\Sigma = \mathbb{RP}_h^2$.

Lemma 1. *Let $\lambda \in \mathbb{R}$ be a real number, U be a linear space of functions on $\Sigma \in \{\mathbb{RP}^2, \mathbb{RP}_h^2\}$ satisfying $\Delta(f) = \lambda f$ and let $n = \sup_{f \in U} \mu(\mathcal{N}(f))$.*

Suppose that $\dim(U) \geq 2n$. For each $x \in \text{Int}(\Sigma)$ consider the set $U_n(x) \subset U$ consisting of eigenfunctions f_x whose nodal graph \mathcal{N}_f contains a vertex x of degree at least $2n$.

Then $\dim(U_n(x)) \geq 1$. Moreover,

- *any f_x has a nodal graph with a unique vertex in $\bar{\Sigma}$;*
- *$\text{ord}(f_x, x) = n$;*
- *faces of $\bar{\mathcal{N}}(f_x)$ are homeomorphic to D^2 ;*
- *$\dim(U_n(x)) \leq 2$.*

Proof. The first inequality is a consequence of Proposition 1.

Now, consider the nodal graph $\bar{\mathcal{N}}(f)$ in $\bar{\Sigma}$ of a function $f \in U_n(x)$ with highest $\text{ord}(f, x)$. Add new edges cutting non-simply-connected faces into discs (and hence making the graph connected). Contract some edges to merge all the vertices with x . We have obtained a new graph $\bar{\mathcal{N}}'$. If some of propositions of Lemma 1 failed, namely, there were other vertices except x , or there were non-simply-connected faces of $\bar{\mathcal{N}}$, or $\text{ord}(f_x, x) > n$, or $\dim(U_n(x)) > 2$, then the degree of vertex x in the new graph satisfies $\deg_{\bar{\mathcal{N}}'}(x) \geq 2n + 2$ and hence $\bar{\mathcal{N}}'$ has at least $n + 1$ edges. The number of faces of $\bar{\mathcal{N}}'$ is the same as for $\bar{\mathcal{N}}$, no more than n . Now Euler characteristic of $\bar{\Sigma} \cong \mathbb{RP}^2$ can be estimated as

$$\begin{aligned} 1 &= \#\{\text{vertices of } \bar{\mathcal{N}}'\} - \#\{\text{edges of } \bar{\mathcal{N}}'\} + \#\{\text{faces of } \bar{\mathcal{N}}'\} \leq \\ &\leq 1 - (n + 1) + n = 0, \end{aligned}$$

but it is a contradiction.

The inequality $\dim(U(x)) \leq 2$ follows now from the fact that, according to the proof of Proposition 1, otherwise there would be a function $f_x \in U_{n+1}(x)$, such that $\text{ord}(f_x, x) > n$ which is already ruled out.

Hence all the ways Lemma 1 could fail, lead to the contradiction and Lemma 1 is proven. \square

Lemma 1 implies that if $\dim(U) \geq 2n$ then for each point $x \in \mathbb{RP}^2$ there is either unique (up to \mathbb{R}^*) eigenfunction f_x whose nodal graph \mathcal{N}_{f_x} has the vertex x of degree $2n$, or a 2-dimensional space $U(x)$ of such functions. We prove now that the case $\dim(U_n(x)) = 2$ described above is impossible in the case of odd n .

Lemma 2. *In the setting of Lemma 1 if n is an odd number then we have $\dim(U_n(x)) = 1$.*

Before we get to the proof of Lemma 2, let us quickly mention a technical property of nodal graphs in the surface $\bar{\Sigma}$ with the collapsed boundary.

Proposition 2. *Suppose $f(p)$ is a continuous family of functions depending on a parameter p , each $f(p)$ satisfy $\Delta f(p) = \lambda f(p)$ for some real number $\lambda \in \mathbb{R}$ and the nodal graph $\bar{\mathcal{N}}(p) = \bar{\mathcal{N}}(f(p))$ has only one vertex $x(p)$ in $\bar{\Sigma}$ and $\deg_{\bar{\mathcal{N}}(p)}(x(p)) = 2n$. Then the loops provided by edges of $\bar{\mathcal{N}}(p)$ do not change their homotopy class in the local system $\pi_1(\bar{\Sigma}, x(p))$ while p moves along $\text{Int}(\Sigma)$.*

The idea of the proof is that since $x(p_0)$ is the only vertex of the graph in $\bar{\Sigma}$, there is no more than two rays of the nodal graph in Σ approaching each connected component of the boundary. Hence for all p close to p_0 these rays can connect only with each other near the collapsed boundary component in $\bar{\Sigma}$. For details, see [2, Proposition 3].

Proof of Lemma 2. Suppose that $\dim(U_n(x)) = 2$. Then, according to Bers theorem, for some polar coordinates (r, θ) near x and polar coordinates (R, ϕ) in $U(x)$ we have the following approximation

$$f_{(R, \phi)}(r, \theta) = Rr^n \cos((n)\theta + \phi) + O(r^n).$$

Hence, taking the star of the function $\sigma(\bullet, x) : PU \rightarrow F_x$ is a diffeomorphism of $PU \cong S^1 \cong F_x$, and the generator loop of PU rotates each ray of $\sigma(f, x)$ to the subsequent one.

Consider the universal cover

$$\begin{aligned} \pi : \mathbb{R} \times [-1, 1] &\xrightarrow{(\bullet/2\mathbb{Z}) \times id} S^1 \times [-1, 1] \cong S^2 \setminus \{\pm x\} \rightarrow \\ &\xrightarrow{\bullet/\{\pm 1\}} \mathbb{R}P^2 \setminus \{x\} \cong S^1 \ltimes [-1, 1] \end{aligned}$$

such that the fundamental group of $\mathbb{R}P^2 \setminus \{x\}$ acts by integer shifts on \mathbb{R} -factor. Then the lift of the edge of $\bar{\mathcal{N}}(f)$ joins $(y, -1)$ and $(y + \frac{j}{n}, e)$ for some $j \in \mathbb{Z}$ and $e \in \{-1, 1\}$. Applying Proposition 2 to an isotopy of a graph induced by a generator loop of PU , we conclude that all the points $(y + \frac{i}{n}, -1)$ are joined with $(y + \frac{i+j}{n}, e)$ by the lift of some edge of $\bar{\mathcal{N}}(f)$. Since this identification is a part of an involution prescribed by the lifts of the edges, e has to be equal to 1. We conclude that $(y + \frac{i}{n}, 1)$ is joined with $(y + \frac{i-j}{n}, -1)$. The lift of the antipodal map on $S^2 \setminus \{\pm x\}$ replaces j defined in this way by $-j$. But the graph, lifted along the quotient by $\{\pm 1\}$ is invariant under the antipodal map, therefore $j = 0$.

But this couldn't be either, as in this case for sufficiently small ε we get

$$\begin{aligned} \operatorname{sgn}(f(\pi(y + \frac{1}{2n}, -(1 - \varepsilon)))) &= \operatorname{sgn}(f(\pi(y + \frac{1}{2n}, 1 - \varepsilon))) = \\ &= -\operatorname{sgn}(f(\pi(y + \frac{n+1}{2n}, 1 - \varepsilon))) = -\operatorname{sgn}(f(\pi(y + \frac{1}{2n}, -(1 - \varepsilon)))) \end{aligned}$$

which is a contradiction. Here for the first equality we use $j = 0$, for the second equality we use the fact that n is odd and hence we have $\operatorname{sgn}(\sin(\frac{\pi}{2})) = -\operatorname{sgn}(\sin(n\pi + \sin(\frac{\pi}{2})))$ and for the third equality we use the invariance of f under the action of $\pi_1(\mathbb{RP}^2 \setminus \{x\})$. \square

Finally, for an odd n we are left with only possibility that for each point $x \in \mathbb{RP}^2$ there is unique (up to \mathbb{R}^*) eigenfunction f_x such that $\operatorname{ord}(f_x, x) = n$ and no functions with $\operatorname{ord}(f_x, x) \geq n + 1$. Note that it completes the proof of the first bound in the Theorem 16 required for the Theorem 2. Indeed, if in the conditions of the Theorem 16 we have $\dim(U) \geq 2n + 1$, then by Proposition 1 we have the inequality $\dim(U_n(x)) \geq 1 + (2n + 1) - 2n = 2$, which contradicts Lemma 2 for odd n .

Therefore Theorem 2 is now proven. \square

3. SECOND BOUND: CLOSED SURFACE CASE

We switch to the case of $\Sigma = \mathbb{RP}^2$. Suppose that there is a counterexample to the Theorem 1, so that there is an eigenspace U_{2l} with $\dim(U_{2l}) \geq 4l + 2$. Then due to Courant nodal domain theorem the assumption $\sup_{f \in U} (\mu(\mathcal{N}(f))) \leq 2l + 1 = n$ of the Lemmas 1 and 2 holds

and for each point $x \in \mathbb{RP}^2$ we get a function f_x , defined up to a constant, such that $\operatorname{ord}(f_x, x) = n$. The stars of the functions f_x form then a smooth section σ of the star fibration $E_{\mathbb{RP}^2}(2n)$ according to the following technical proposition.

Proposition 3. *Let U be a finite-dimensional eigenspace of the Laplacian. For every $x \in \operatorname{Int}(\Sigma)$ consider the subspace $[f_x] \subset U$ of functions f_x with $\operatorname{ord}(f_x, x) \geq n$. Suppose that for any $x \in \operatorname{Int}(\Sigma)$ the subspace $[f_x]$ is of dimension 1 (i. e. nonzero f_x are defined up to \mathbb{R}^*), and that $\operatorname{ord}(f_x, x) = n$. Define a section $\sigma(x) = s(\mathcal{N}(f_x), x)$. Then this section $\sigma \in \Gamma(E_{\operatorname{Int}(\Sigma)}(2n))$ is smooth.*

The proof follows directly from inverse function theorem, see the paper [2, Proposition 1].

Now in order to achieve the final contradiction let us pull σ back to the star fibration $E_{\mathbb{S}^2}(2n)$ via the universal cover $\mathbb{S}^2 \rightarrow \mathbb{RP}^2$ and

observe that the Euler class $e(E_{\mathbb{S}^2}(2n)) = 2ne(T\mathbb{S}^2) = 4n$ is non-zero and hence $E_{\mathbb{S}^2}(2n)$ can not have a continuous section. Hence, the initial assumption that the multiplicity of λ_{2l} is at least $4l + 2$ is false and thus the multiplicity is at most $4k + 1$.

Unfortunately, we did not find any evident way to make the analogous final consideration with the Euler class in the case $\partial\Sigma \neq \emptyset$, since the surface with boundary has no such class. However, it would be possible to use the relative Euler class if we could put some restrictions on the behavior of the star section σ near the boundary. In the case of a surface M of positive genus such restrictions come from the fact that some loops of nodal graph represent a (constant) non-zero class in $H_1(\tilde{M})$ for orienting cover \tilde{M} . This consideration, based on the method from the paper [14], is the crucial argument in the paper [2]. But in our case of \mathbb{RP}^2 the orienting cover \mathbb{S}^2 has zero homology in dimension 1 and there is no apparent topological obstructions for the nodal graphs in the consideration to exist.

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